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L^2-stability analysis of novel ETD scheme for Kuramoto–Sivashinsky equations

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ABSTRACT

The aim of this paper is to study the stability analysis of novel ETD scheme proposed by the authors based on spectral methods, the exponential time differencing and Taylor expansion. Stability issue of the proposed numerical scheme is related to an analysis of the stability of the corresponding ODE system for time marching approach. It is proved that the novel scheme is L^2-stable in solving the Kuramoto–Sivashinsky model problems. The truncation error and the stability region for the novel scheme are provided. Comparisons with available literature are made.

1. Introduction

For many unsteady partial differential equations, spectral methods have been shown to provide remarkably effective spatial discretization [1]. The application of these discretization methods often leads to systems of stiff ordinary differential equations (ODEs) in time and thus makes the efficient, stable and time integration scheme very essential. Methods of solving these stiff systems have been studied in the literature [2], including, for instance, implicit–explicit methods [3–6], semi-implicit methods [1,7,8], time-splitting methods, projection methods [9,10], integrating factor [11,12] and the exponential time differencing [13–15]. Stability properties for ETD schemes at first proposed in [14]. Stability properties for the first order ETD scheme were shown by Du for Laplace operator [16]. More details for stability of the ETDRK2, ETDRK3 and ETDRK4 can be found in [17]. In this paper, our interest is to study the stability properties of two novel ETD schemes and their modifications which have been shown to perform extremely well in solving various one-dimensional dispersive and dissipative type problems [18].

2. Novel exponential time differencing

Consider the partial differential equation in $[0, 2\pi]$ for time $t > 0$ as

$$u_t = -u_{xx} - \nu u_{xxxx} - uu_x, \quad x \in [0, 2\pi],\ t > 0. \tag{1}$$

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This is called Kuramoto–Sivashinsky (KS) problem. A related linearized equation is as follows

$$u_t = -u_{xx} - \nu u_{xxxx} + \lambda u, \tag{2}$$

where λ is a constant or a specific function. Here λu is a linear representation of the nonlinear term in Eq. (1). Moreover in most of our discussion, initial boundary value problems with periodic boundary conditions are considered. Discretization Eq. (1) in the x variable by Fourier Galerkin spectral approximation leads to a system of the following ODEs

$$\begin{cases} \hat{U}_t = D\hat{U} + N(\hat{U}, t), \quad t > 0, \\ \hat{U}(t = 0) = \hat{U}_0, \end{cases} \tag{3}$$

where D is a diagonal matrix in the form

$$D = \mathrm{diag}(d_1, \dots, d_M), \quad d_k = (k - M/2 - 1)^2 - \nu(k - M/2 - 1)^4, \tag{4}$$

for $k = 1, \dots, M$. $\hat{U}$ is the Fourier coefficients vector and $N(\hat{U}, t)$ is the discretized operator in x variable related to the nonlinear part of Eq. (1). Hereafter we focus on a fixed spatial mesh to provide the approximate solution of (1) by solving the discretized system (3). After some manipulation for $h = t_{n+1} - t_n$ we have

$$\hat{U}_{n+1} = \mathrm{e}^{Dh}\hat{U}_n + \mathrm{e}^{Dh}\int_0^h \mathrm{e}^{-D\eta} N(\hat{U}(t_n + \eta), t_n + \eta)\mathrm{d}\eta. \tag{5}$$

Thus

$$\hat{U}_{n+1} = \mathrm{e}^{Dh}\hat{U}_n + \alpha N(\hat{U}(t_n), t_n) + \beta \left.\frac{\partial N(\hat{U}(t), t)}{\partial t}\right|_{t=t_n} + \gamma \left[N(a_{n+1}, t_n + h) - N(\hat{U}(t_n), t_n) - h\left.\frac{\partial N(\hat{U}(t), t)}{\partial t}\right|_{t=t_n} \right], \tag{6}$$

where

$$a_{n+1} = \mathrm{e}^{Dh}\hat{U}_n + \frac{\mathrm{e}^{Dh} - 1}{D} N(\hat{U}(t_n), t_n) + \frac{\mathrm{e}^{Dh} - 1 - Dh}{D^2} \left.\frac{\partial N(\hat{U}(t), t)}{\partial t}\right|_{t=t_n}, \tag{7}$$

$$\alpha = D^{-1}(\mathrm{e}^{Dh} - I), \tag{8}$$

$$\beta = -hD^{-1} + D^{-2}(\mathrm{e}^{Dh} - I), \tag{9}$$

and

$$\gamma = -h^2 D^{-1} - 2hD^{-2} + 2D^{-3}(\mathrm{e}^{Dh} - I). \tag{10}$$

The numerical scheme in the form (6) suffers numerical instability [18]. Vaissmoradi et al. [18] explained how to compute α, β and γ successfully, using complex contour integral.

3. L^2-stability analysis

In this section we discuss the asymptotic stability properties of the first order ETD and the two novel ETD schemes (7) and (6) for KS model problems.

3.1. First order ETD scheme

The result of applying the first ETD scheme to the general nonlinear PDEs with Laplace operator in their linear term and with no spatial approximation is introduced by Du and Zhu in the year 2004, [16]. Here we extend their idea to KS problem, when the fourth order operator appears in its linear term. Direct applying the first order ETD scheme at time step t_n for the solution u_n is equivalent to solving

$$\begin{cases} \dfrac{\partial}{\partial t} w(x, t) = \left(-\dfrac{\partial^2}{\partial x^2} - \nu \dfrac{\partial^4}{\partial x^4}\right) w(x, t) + \mathcal{N}(u_n(x), x, t), \\ w(x, 0) = u_n(x), \end{cases} \tag{11}$$

where $w(\cdot, \Delta t) = u_{n+1}$ and $t_{n+1} = t_n + \Delta t$. Corresponding to Eq. (2), we can reduce Eq. (11) to the following form

$$\begin{cases} \dfrac{\partial}{\partial t} w(x, t) = \left(-\dfrac{\partial^2}{\partial x^2} - \nu \dfrac{\partial^4}{\partial x^4}\right) w(x, t) + \lambda u_n(x), \\ w(x, 0) = u_n(x). \end{cases} \tag{12}$$

We now continue with our a stability theorem concerning (12). Let λ and $\nu \geq 0$ be some complex and real constants respectively. By using the spectral Galerkin method, the stability issue is related to an analysis of the stability of the corresponding system of ODEs.

Theorem 1. *For the Fourier–Galerkin scheme of (12) with periodic boundary conditions, if λ and $\nu \geq 0$ are specific constants then the following statements are hold.*

(i) *The first order ETD scheme is unconditionally asymptotically L^2-stable for $|\lambda| = 0$.*
(ii) *The first order ETD scheme is asymptotically L^2-stable iff $\Re\lambda < 0$ and $\tau \leq \frac{-2\Re\lambda}{|\lambda|^2}$.*

Proof. By using Fourier Galerkin scheme we can write

$$u_n(x) = \sum_{k\in Z} \hat{u}_{n,k} e^{ikx}, \tag{13}$$

$$w(x,t) = \sum_{k\in Z} \hat{w}_k(t) e^{ikx}, \tag{14}$$

in which

$$\hat{u}_{n,k} = \frac{1}{2\pi} \int_0^{2\pi} u_n(x) e^{-ikx} dx \quad k \in Z.$$

Substituting Eqs. (13) and (14) into (12), we have

$$\begin{cases} \dfrac{d}{dt} \hat{w}_k(t) = m_k \hat{w}_k(t) + \lambda \hat{u}_{n,k}, \\ \hat{w}_k(0) = \hat{u}_{n,k}, \end{cases} \tag{15}$$

where $m_k = |k|^2 - \nu |k|^4$. The exact solution of above system of ODEs is:

$$\hat{w}_k(\tau) = e^{m_k \tau} \hat{u}_{n,k} + \lambda \int_0^{\tau} e^{m_k(\tau - s)} \hat{u}_{n,k} ds. \tag{16}$$

For any $k \in Z$, we define

$$h_k(\tau) = \begin{cases} 1 + \lambda\tau, & m_k = 0, \\ 1 + \dfrac{1 - e^{m_k\tau}}{m_k}(-\lambda - m_k), & m_k \neq 0. \end{cases} \tag{17}$$

From Eqs. (16) and (17) we see that $\hat{w}_k(\tau) = h_k(\tau)\hat{u}_{n,k}$. Thus

$$w(x,\tau) = \sum_{k\in Z} h_k(\tau)\hat{u}_{n,k} e^{ikx} = \frac{1}{2\pi} \int_0^{2\pi} \sum_{k\in Z} h_k(\tau) u_n(y) e^{ik(x-y)} dy \tag{18}$$

and $\|w(\cdot,\tau)\|^2_{L^2(0,2\pi)} = \sum_{k\in Z} |h_k|^2 |\hat{u}_{n,k}|^2$, where $w(\cdot,\tau) = u_{n+1}$. We continue by using $|h_k(\tau)| \leq 1$ for each $k \in Z$ to obtain the asymptotic L^2-stability.

$$|1 + \lambda\tau| \leq 1 \quad m_k = 0, \tag{19}$$

$$\left| 1 + \frac{1 - e^{m_k\tau}}{m_k}(-\lambda - m_k) \right| \leq 1 \quad m_k < 0. \tag{20}$$

The inequalities (19) and (20) hold when $|\lambda| = 0$. This finishes the proof of (i). For $|\lambda| \neq 0$, there exist two cases for $m_k = 0$ and $m_k < 0$. Suppose that the first order ETD scheme is asymptotically L^2-stable when $m_k = 0$ inequality (19) is equivalent to $\tau(2\Re\lambda + \tau|\lambda|^2) \leq 0$. Therefore one can get $\tau \leq \hat{\tau}_0 = -\frac{2\Re\lambda}{|\lambda|^2}$. Thus $\Re\lambda < 0$. For $m_k < 0$ we consider the inequality (20). This inequality equivalent to

$$(1 + r_k(\lambda + m_k))(1 + r_k(\bar{\lambda} + m_k)) \leq 1, \tag{21}$$

where $r_k = \frac{1 - e^{m_k\tau}}{-m_k} > 0$. Thus

$$r_k \left[2\Re\lambda + 2m_k + r_k^2(|\lambda|^2 + 2m_k\Re\lambda + m_k^2) \right] \leq 0,$$

or

$$2(\Re\lambda + m_k) + r_k(|\lambda|^2 + 2m_k\Re\lambda + m_k^2) \leq 0. \tag{22}$$

If $m_k \leq -|\lambda|$, the inequality (22) holds since $\Re\lambda \leq |\lambda| \leq -m_k$ then

$$\begin{aligned} 2(\Re\lambda + m_k) + r_k(|\lambda|^2 + 2m_k\Re\lambda + m_k^2) &\leq 2(\Re\lambda + m_k) + r_k(2m_k\Re\lambda + 2m_k^2) \\ &= 2(\Re\lambda + m_k)(1 + m_k r_k) = 2(\Re\lambda + m_k)e^{m_k\tau} \leq 0. \end{aligned} \tag{23}$$

For $m_k > -|\lambda|$ the inequality (22) can be written as

$$2(\Re\lambda + m_k) + \frac{1 - e^{m_k\tau}}{-m_k}(|\lambda|^2 + 2m_k\Re\lambda + m_k^2) \leq 0. \tag{24}$$

Let us to define $g = |\lambda|^2 + 2m_k\Re\lambda + m_k^2$, then

$$\frac{g - ge^{m_k\tau} - 2m_k\Re\lambda - 2m_k^2}{-m_k} \leq 0. \tag{25}$$

Since $m_k < 0$, we must have $g - ge^{m_k\tau} - 2m_k\Re\lambda - 2m_k^2 \leq 0$. Thus

$$|\lambda|^2 - m_k^2 \leq ge^{m_k\tau}. \tag{26}$$

Note that for $m_k > -|\lambda|$, it is easy shows that $g > 0$. Now from inequality (26) we have

$$e^{m_k\tau} \geq \frac{|\lambda|^2 - m_k^2}{g},$$

hence

$$m_k\tau \geq \ln\left(\frac{|\lambda|^2 - m_k^2}{g}\right),$$

thus

$$\tau \leq \tau_k = \frac{1}{-m_k}\ln\left(\frac{|\lambda|^2 + 2m_k\Re\lambda + m_k^2}{|\lambda|^2 - m_k^2}\right). \tag{27}$$

Let us define

$$f(m_k) = \frac{1}{-m_k}\ln\left(\frac{|\lambda|^2 + 2m_k\Re\lambda + m_k^2}{|\lambda|^2 - m_k^2}\right).$$

If we prove that $\mathrm{Min} f(m_k) = \frac{-2\Re\lambda}{|\lambda|^2}$, this finishes the proof of one side of case (ii). The other side can be proved conversely. For simplicity suppose that $b = |\lambda|$, $a = -\Re\lambda$ and $x = -m_k$, hence

$$f(x) = \frac{1}{x}\ln\left(\frac{b^2 + 2ax + x^2}{b^2 - x^2}\right), \quad b > x > 0,\ b > a. \tag{28}$$

First we show that $f(x)$ is an increasing function. To do this we show that $f'(x) > 0$. For $g_1(x) = \ln\left(\frac{b^2+2ax+x^2}{b^2-x^2}\right)$ we have $f(x) = \frac{1}{x}g_1(x)$ and $g_1'(x) = \frac{2x+2a}{x^2+b^2+2ax} + \frac{2x}{b^2-x^2}$, therefore

$$f'(x) = \frac{-1}{x^2}g_1(x) + \frac{1}{x}g_1'(x).$$

The condition $f'(x) > 0$ is equivalent to $xg_1'(x) - g_1(x) > 0$. For $T(x) = xg_1'(x) - g_1(x)$ we have

$$\begin{aligned} T'(x) &= xg_1''(x) \\ &= x\frac{ax^5 + (3b^2 + a^2)x^4 + 2ab^2x^3 + 4a^2b^2x^2 + ab^4x + b^4(b^2 - a^2)}{(x^2 + b^2 + 2ax)(b^2 - x^2)^2}. \end{aligned}$$

It is obvious that $T'(x) > 0$. Thus $T(x)$ is an increasing function, i.e., $T(x) > T(0) = 0$. Then $f'(x) > 0$ and $f(x)$ is an increasing function. Therefore $\tau \leq \tau_0 = \frac{2a}{b^2} = -\frac{2\Re\lambda}{|\lambda|^2} \leq \tau_k$. □

3.2. Second order novel ETD scheme

In this subsection we study the asymptotical L^2-stability analysis for the second order novel ETD scheme.

Theorem 2. *For the second order method that is proposed in Section 2 for Eq. (2) with periodic boundary condition and constant coefficient $\lambda \leq 0$, if $h \leq \frac{-1}{\lambda}$ then second order novel ETD scheme is asymptotically L^2-stable.*

Proof. The second order novel ETD scheme can be written as

$$u_{n+1} = \mathrm{e}^{Dh}u_n + D^{-1}(\mathrm{e}^{Dh} - I)N(u(t_n), t_n) + (-hD^{-1} + D^{-2}(\mathrm{e}^{Dh} - I))\frac{\partial N(u(t), t)}{\partial t}\bigg|_{t=t_n}. \tag{29}$$

The application of this scheme for Eq. (2) is equivalent to

$$u_{n+1} = (\mathrm{e}^{Dh} + D^{-1}(\mathrm{e}^{Dh} - I)\lambda + (-hD^{-1} + D^{-2}(\mathrm{e}^{Dh} - I))\lambda(D + \lambda))u_n. \tag{30}$$

Let $y = -h\lambda$ and $\xi = h(\nu|k|^4 - |k|^2)$, with k being the Fourier modes, the corresponding amplification factor is

$$g(\xi, y) = \begin{cases} 1 - y + \dfrac{y^2}{2}, & \xi = 0, \\ \mathrm{e}^{-\xi} + \left(1 + 2\dfrac{\mathrm{e}^{-\xi} - 1}{\xi}\right)y + \dfrac{\xi - 1 + \mathrm{e}^{-\xi}}{\xi^2}y^2, & \xi > 0. \end{cases} \tag{31}$$

The asymptotically L^2-stability condition is

$$|g(\xi, y)| \le 1, \quad \text{or} \quad -1 \le g(\xi, y) \le 1. \tag{32}$$

Rewriting (31) gives

$$g(\xi, y) = \begin{cases} 1 - y + \dfrac{y^2}{2}, & \xi = 0, \\ (\mathrm{e}^{-\xi}(y + \xi)^2 + y\xi^2 - 2y\xi + y^2\xi - y^2)/\xi^2, & \xi > 0. \end{cases} \tag{33}$$

In the case $\xi = 0$, for $y \in [0, 2]$ the relation (32) holds. Now we try to answer that for $\xi > 0$, for what interval $y \in [0, \alpha]$ the relation $g(\xi, y) < 1$ holds. For $\xi > 0$ the relation $g(\xi, y) < 1$ is equivalent to

$$(y + \xi)((\mathrm{e}^{-\xi} - 1)(y + \xi) + y\xi) \le 0. \tag{34}$$

Thus

$$y \le \frac{\xi(\mathrm{e}^{\xi} - 1)}{\xi \mathrm{e}^{\xi} - \mathrm{e}^{\xi} + 1}. \tag{35}$$

Then for each $\xi \ge 0$, y belongs to interval $\left[0, \frac{\xi(\mathrm{e}^{\xi}-1)}{\xi \mathrm{e}^{\xi} - \mathrm{e}^{\xi} + 1}\right]$. If ξ tends to zero then $y \in [0, 2]$ and ξ tends to infinity $y \in [0, 1]$. Therefore the final interval for y in order to satisfying the relation $g(\xi, y) < 1$ for each $\xi \ge 0$ is $[0, 1]$. The left part of the inequality (32) is $g(\xi, y) \ge -1$. We show that this inequality for each $\xi \ge 0$ and $y \in [0, 1]$ holds. Considering relation (33) for $\xi > 0$ the inequality $g(\xi, y) \ge -1$ is equivalent to

$$(\mathrm{e}^{-\xi} - 1 + \xi)y^2 + (-2\xi + \xi^2 + 2\xi \mathrm{e}^{-\xi})y + \xi^2(1 + \mathrm{e}^{-\xi}) \ge 0. \tag{36}$$

For $\xi \ge 1$ the functions $T_0(\xi) = \xi^2(1 + \mathrm{e}^{-\xi})$, $T_1(\xi) = -2\xi + \xi^2 + 2\xi \mathrm{e}^{-\xi}$ and $T_2(\xi) = \mathrm{e}^{-\xi} - 1 + \xi$ are increasing. Then we have

$$(\mathrm{e}^{-\xi} - 1 + \xi)y^2 + (-2\xi + \xi^2 + 2\xi \mathrm{e}^{-\xi})y + \xi^2(1 + \mathrm{e}^{-\xi}) \ge \mathrm{e}^{-1}y^2 + (2\mathrm{e}^{-1} - 1)y + (1 + \mathrm{e}^{-1}) > 0. \tag{37}$$

The relation (37) shows that the inequality $g(\xi, y) \ge -1$ holds for each $\xi \ge 1$. Now for $0 \le \xi < 1$ we show that $(T_1(\xi))^2 - 4T_2(\xi)T_0(\xi) \le 0$. i.e.

$$8\mathrm{e}^{-\xi} - 8 + 8\xi - \xi^2 \ge 0. \tag{38}$$

For simplicity let $k(\xi) = 8\mathrm{e}^{-\xi} - 8 + 8\xi - \xi^2$, then $k'(\xi) = -8\mathrm{e}^{-\xi} + 8 - 2\xi$. For $M(\xi) = -8\mathrm{e}^{-\xi} + 8 - 2\xi$, we have $M'(\xi) = 8\mathrm{e}^{-\xi} - 2 \ge 0$ for $0 \le \xi < 1$. This means that $M(\xi)$ is an increasing function therefore $k(\xi)$ is an increasing function too. This shows that relation (38) holds and the theorem is proved. □

3.3. Fourth order novel ETD scheme

The aim of this subsection is to study the stability properties of the fourth order novel ETD scheme that is described in Section 2.

Theorem 3. *For the fourth order scheme that is proposed in Section 2 for Eq. (2) with periodic boundary conditions and constant coefficient $\lambda \le 0$, if $h \le \frac{-2.5}{\lambda}$ then it is asymptotically L^2-stable.*

Proof. The fourth order novel ETD scheme is

$$u_{n+1} = e^{Dh}u_n + \frac{e^{Dh}-1}{L}N(u(t_n), t_n) + \frac{e^{Dh}-1-Dh}{D^2}\left.\frac{\partial N(u(t),t)}{\partial t}\right|_{t=t_n} + \frac{2e^{Dh}-2-2Dh-(Dh)^2}{D^3h^2}\left[N(a_{n+1}, t_n+h) - N(u(t_n), t_n) - h\left.\frac{\partial N(u(t),t)}{\partial t}\right|_{t=t_n}\right]. \tag{39}$$

Let $y = -h\lambda$ and $\xi = h(\nu|k|^4 - |k|^2)$, with k being the Fourier modes, the corresponding amplification factor is

$$q_3(\xi, y) = \begin{cases} 1 - y + \frac{y^2}{2} - \frac{y^3}{6}, & \xi = 0 \\ h_0(\xi) + h_1(\xi)y + h_2(\xi)y^2 + h_3(\xi)y^3, & \xi > 0 \end{cases} \tag{40}$$

in which

$$h_0(\xi) = e^{-\xi}, \qquad h_1(\xi) = \xi g_2 + g_1 - h_0 f_2 + f_2 - \xi f_2,$$
$$h_2(\xi) = g_2 - f_2(\xi g_2 + g_1) - f_2, \qquad h_3(\xi) = -f_2 g_2$$

where

$$g_1(\xi) = \frac{e^{-\xi}-1}{\xi}, \qquad g_2(\xi) = \frac{e^{-\xi}-1-\xi}{\xi^2}, \qquad f_2(\xi) = -\frac{2e^{-\xi}-2-2\xi-\xi^2}{\xi^3}.$$

To prove the asymptotically $Ł^2$-stability for Eq. (39), we must show that for which interval for y, the relation $-1 \le q_3(\xi, y) \le 1$ holds for h $\xi \ge 0$. First we show that for each $\xi \ge 0$ and $y \ge 0$ the inequality $q_3(\xi, y) \le 1$ holds. Lets $R_3 = q_3(\xi, y) - 1$ thus

$$R_3(\xi, y) = \begin{cases} y\left(-1 + \frac{y}{2} - \frac{y^2}{6}\right), & \xi = 0, \\ h_0(\xi) - 1 + h_1(\xi)y + h_2(\xi)y^2 + h_3(\xi)y^3, & \xi > 0. \end{cases} \tag{41}$$

It is easy to show that for

$$R_2(\xi, y) = \begin{cases} -1 + \frac{y}{2} - \frac{y^2}{6}, & \xi = 0, \\ \xi^2 h_3(\xi) - \xi h_2 + h_1(\xi) + [-\xi h_3(\xi) + h_2(\xi)]y + h_3(\xi)y^2, & \xi > 0, \end{cases} \tag{42}$$

one can write $R_3(\xi, y) = (y + \xi)R_2(\xi, y)$.

It is trivial that $R_2(\xi, y) \le 0$. Then the inequality $q_3(\xi, y) \le 1$ holds. Another side of the inequality is $-1 \le q_3(\xi, y)$. By defining $Y_3(\xi, y) = q_3(\xi, y) + 1$ we have

$$Y_3(\xi, y) = \begin{cases} 2 - y + \frac{y^2}{2} - \frac{y^3}{6}, & \xi = 0 \\ h_0(\xi) + 1 + h_1(\xi)y + h_2(\xi)y^2 + h_3(\xi)y^3 & \xi > 0. \end{cases} \tag{43}$$

Minimum $Y_3(\xi, y)$ for $\xi \in [0, \infty)$ and $y \in [0, 2.5]$ happens at ($\xi = 0.4059539203747227, y = 2.5$), where $\mathrm{Min}Y_3(\xi, y) = 0.000957774372333$. This proves the Theorem 3. □

4. Truncation error analysis

To study the local truncation error of the novel ETD scheme (39) we consider the following equation, which comes from linearization of Eq. (3):

$$u_t = \xi u + \lambda u. \tag{44}$$

The exact solution for Eq. (44) is $u(t) = e^{(\xi+\lambda)t}$. Taylor expansion gives

$$u(t) = 1 + (\xi+\lambda)t + \frac{1}{2}(\xi+\lambda)^2t^2 + \frac{1}{6}(\xi+\lambda)^3t^3 + \frac{1}{24}(\xi+\lambda)^4t^4 + O(t^5). \tag{45}$$

The corresponding amplification factor for Eq. (39) with respect to Eq. (44) is in the form

$$g(\xi, t) = g_1(\xi, t) + \frac{2e^{\xi t} - 2 - 2\xi t - (\xi t)^2}{\xi^3 t^2}(\lambda g_1(\xi, t) - \lambda - \lambda t(\lambda + \xi)), \tag{46}$$

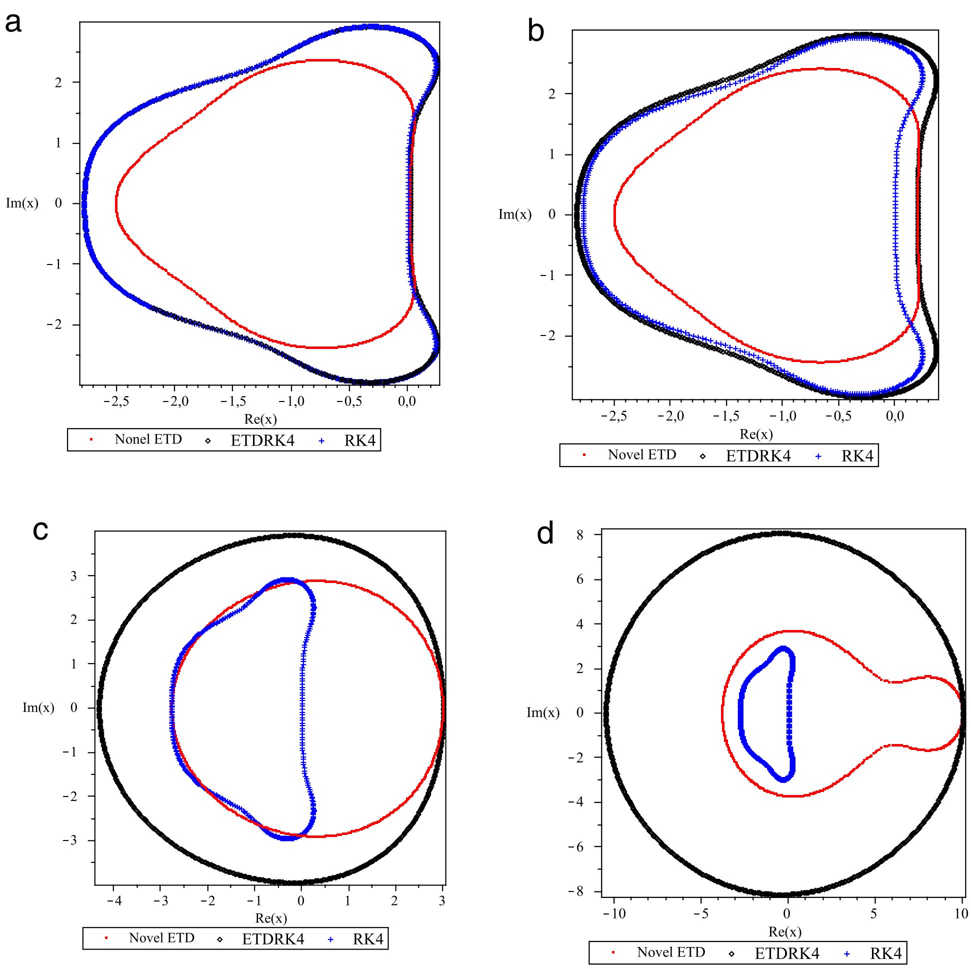

Fig. 1. Stability regions for three schemes in the complex planes of $x = \lambda t$ for different values of $y = \xi t$. (a) $y = -0.02$, (b) $y = -0.2$, (c) $y = -3$ and (d) $y = -10$.

where

$$g_1(\xi, t) = e^{\xi t} + \frac{e^{\xi t} - 1}{\xi}\lambda + \frac{e^{\xi t} - 1 - \xi t}{\xi^2}\lambda(\lambda + \xi).$$

By using Taylor series expansion for $g(\xi, t)$ in t gives

$$g(\xi, t) = 1 + (\lambda + \xi)t + (\lambda + \xi)^2\frac{t^2}{2} + (\lambda + \xi)^3\frac{t^3}{6} + \frac{1}{72}\xi(\lambda + \xi)^2(7\lambda + 3\xi)t^4 + O(t^5). \tag{47}$$

Therefore, in general, without complex integration and by comparison between Eqs. (45) and (47) the local truncation error of the novel ETD scheme (39) is $O((\Delta t)^4)$.

5. Stability region

We define the stability regions as the parameter regions such that the absolute magnitude of the amplification factor being less than or equal to 1 when applied to the Eq. (44). In the following we discuss the stability regions for schemes ETDRK4 [14,17] and RK4 [19] against novel ETD scheme (39) in the complex plane of $x = \lambda t$ for different values of $y = \xi t$. Fig. 1 shows the stability regions for $y = \xi t = -0.02, -0.2, -3, -10$ for three different schemes. In Fig. 1(a) and (b) we show that the region of stability for novel ETD scheme is smaller than the stability regions of ETDRK4 and RK4 for $y = -0.02$ and $y = -0.2$ respectively. Fig. 1(c) and (d) show that for small y the stability region of novel ETD scheme is larger than stability region of RK4 while the region of stability for both schemes are smaller than ETDRK4 scheme.

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